Circumnavigation Using Distance Measurements (Extended Version)

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***Abstract*— Consider a stationary agent A at an unknown location and a mobile agent B that must move to the vicinity of and then circumnavigate A at a prescribed distance from A. In doing so, B can only measure its distance from A, and knows its own position in some reference frame. This paper considers this problem, which has applications to surveillance or maintaining an orbit. In many of these applications it is difficult for B to directly sense the location of A, e.g. when all that B can sense is the intensity of a signal emitted by A. This intensity does, however provide a measure of the distance. We propose a nonlinear periodic continuous time control law that achieves the objective. Fundamentally, B must exploit its motion to estimate the location of A, and use its best instantaneous estimate of where A resides, to move itself to achieve the ultimate circumnavigation objective. The control law we propose marries these dual goals and is globally exponentially convergent. We show through simulations that it also permits B to approximately achieve this objective when A experiences slow, persistent and potentially nontrivial drift.**

I. Introduction

In surveillance missions the main objective is to obtain information about the target of interest by monitoring it for a period of time. In most cases it is desirable to monitor the target by circumnavigating it from a prescribed distance. In recent years this problem has been addressed in the context of autonomous agents, where an agent or a group of agents accomplish the surveillance task. This problem has been extensively studied for the case where the position of the target is known and the agent(s) can measure specific information about the source, such as distance, power, angle of arrival, time difference of arrival, etc. See [1], [2], [3] and references therein. However, in many situations, knowing the position of the target is not practical, e.g. if one wants to find and monitor an unknown source of an electromagnetic signal. This paper addresses the problem where the position of the source is unknown, only one agent is involved, and the only information continuously available to the agent is its own position and its distance (not relative position) from the source.

There are other recent papers that consider a problem related to the one addressed in this paper. For instance [4] and [5] using concepts from switched adaptive control, consider the case where an agent must move itself to a point at a pre-set distance from three sources with unknown position in the plane using distance measurements. Another related paper is [6], where the agent's objective is to estimate the position of the source using distance measurements only.

The problem addressed in this paper can be studied as a *dual control* problem as well. In a dual control problem the aim is identify the unknown parameters of the system and achieve a control objective simultaneously [7]. In this case the mobile agent must estimate the location of the target from the distance measurements, and use this estimate to execute its control law for achieving its objective of circumnavigating the target from a prescribed distance.

To estimate the target location the agent must move in a trajectory that is not confined to a straight line in two dimensions and to a plane in three dimensions. To perform robust estimation this avoidance of collinear/coplanar motion must be *persistent* in a sense described in [6]. A key feature of the circumnavigation objective is that it ensures this requirement thus aiding the estimation task.

The structure of the paper is as follows. In the next section the problem is formally defined, and in Section III the proposed algorithm is introduced. The analysis of the control laws is presented in Section IV. In Section V the persistent excitation condition on the signals is established, and The proof of exponential stability of the system is provided. In Section VI a method to choose one of the parameters in the control laws is presented. The simulation results are presented in Section VII. In the final section future directions and concluding remarks are presented.

II. Problem Statement

In what follows we formally define the problem addressed in this paper and introduce relevant assumptions.

Problem 2.1: Consider a source at an unknown constant position x and an agent at $y(t)$ in $\mathbb{R}^n$ ($n \in \{2,3\}$) at time $t \in [0,\infty]$. Knowing $y(t)$, a desired distance d, and the measurement

$$D(t) = \|y(t) - x\| \tag{II.1}$$

find a control law that ensures that asymptotically, $y(t)$ moves on a trajectory at a distance d from x.

Here as in the rest of the paper $\|.\|$ denotes 2-norm. For convenience in the rest of the paper, we impose the following constraint:

Constraint 2.1: The agent trajectory $y(t) : \mathbb{R} \mapsto \mathbb{R}^n$, is to be twice differentiable. Further, there exists $M_0 > 0$, such that $\forall t \in \mathbb{R} : \quad \|y(t)\| + \|\dot{y}(t)\| + \|\ddot{y}(t)\| \leq M_0$.

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This constraint ensures that the motion of the agent can be executed with finite force. One can break down the problem into the following two sub-problems:

1) How one can estimate x?
2) How one can make the agent move on a trajectory at a distance d from x, which, in a sense to be made precise in the sequel, persistently spans $\mathbb{R}^n$?

The first sub-problem is addressed in [6]. However, for the algorithm of [6] to work one requires that $\forall t \in \mathbb{R}$,

$$\alpha_1 I \leq \int_t^{t+T_1} \dot{y}(\tau)\dot{y}(\tau)^\top d\tau \leq \alpha_2 I$$

for some α_1,α_2, and T_1 which are strictly positive. This condition is the well-known persistent excitation (p.e.) condition. This condition requires that the agent in n-dimensional space must persistently avoid a trajectory that is confined to the vicinity of a single $(n-1)$-dimensional hyperplane. As noted earlier, the circumnavigation objective in fact assists in maintaining p.e.

III. Proposed Algorithm

We first present an algorithm for estimating x. To this end for $\alpha > 0$ generate

$$\eta(t) = \dot{z}_1(t) = -\alpha z_1(t) + \frac{1}{2}D^2(t), \tag{III.1}$$

$$m(t) = \dot{z}_2(t) = -\alpha z_2(t) + \frac{1}{2}y^\top(t)y(t), \tag{III.2}$$

$$V(t) = \dot{z}_3(t) = -\alpha z_3(t) + y(t), \tag{III.3}$$

where $z_1(0)$, and $z_2(0) = 0$ are arbitrary scalars, and $z_3(0)$ is an arbitrary vector. Note that the generation of $\eta(t)$, $m(t)$ and $V(t)$ requires simply the measurements $D(t)$ and the knowledge of the localizing agent's own position, and can be performed without explicit differentiation.

Define now the estimator:

$$\dot{\hat{x}}(t) = -\gamma V(t)(\eta(t) - m(t) + V^\top(t)\hat{x}(t)), \tag{III.4}$$

where $\hat{x}(t)$ denotes the estimate of x at time t, and $\gamma > 0$ is the adaptive gain. This estimator is in fact identical to the one presented in [6]. Later, we shall show that under suitable conditions $\hat{x}$ approaches x.

Define

$$\hat{D}(t) = \|y(t) - \hat{x}(t)\|, \tag{III.5}$$

and the control law

$$\dot{y}(t) = \dot{\hat{x}}(t) - \left[(\hat{D}^2(t) - d^2)I - A(t)\right](y(t) - \hat{x}(t)), \tag{III.6}$$

where $A(\cdot) : \mathbb{R} \to \mathbb{R}^{n\times n}$ obeys four conditions captured in the assumption below. As will be proved in the next section, this control law aims at moving the agent so that $\hat{D}$ converges to d, i.e. the agent takes up the correct distance from the estimated of the source position, $\hat{x}$. If also $\hat{x}$ converges to x, then D converges to $\hat{D}$, hence D converges to d.

Assumption 3.1: (i) There exists a $T > 0$ such that for all t,

$$A(t + T) = A(t). \tag{III.7}$$

(ii) $A(t)$ is skew symmetric for all t.
(iii) $A(t)$ is differentiable everywhere.
(iv) the derivative of the solution of the differential equation below is persistently spanning.

$$\dot{y}^*(t) = A(t)y^*(t). \tag{III.8}$$

for any arbitrary nonzero value of $y^*(0)$. More precisely, there exists a $T_1 > 0$, and $\alpha_i > 0$ such that for all $t \geq 0$ there holds

$$\alpha_1\|y^*(t)\|^2 I \leq \int_t^{t+T_1} \dot{y}^*(\tau)\dot{y}^*(\tau)^\top d\tau \leq \alpha_2\|y^*(t)\|^2 I. \tag{III.9}$$

A consequence of the fact that $A(t)$ is skew symmetric is that for all $\nu \in \mathbb{R}^n$ and $t \geq 0$

$$\nu^\top A(t)\nu = 0. \tag{III.10}$$

We note that the results below hold even if $A(t)$ were permitted to lose differentiability at a countable number of points. However, that would imply that $\dot{y}$ would lose differentiability at these same points, resulting in the physically unappealing need for an impulsive force to act on $y(t)$.

As will be shown in Section VI, in two dimensions, with E the rotation matrix,

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{III.11}$$

and a any real scalar, it suffices to chose $A(t)$ as the constant matrix aE. In three dimensions however, the selection of $A(t)$ is more complicated. Specifically for (III.9) to hold with a constant A, A must be nonsingular. No 3×3 skew symmetric matrix is however nonsingular, leading to the need for a periodic $A(t)$, whose selection is described in Section VI. To summarize, the overall system is described by (II.1) - (III.6), with z_i, $\hat{x}$, and y, serving as the underlying state variables.

IV. Analysis

We shall begin by establishing that the quantities $\eta(t)$ and $m(t)$, computable from the measurements as described in the previous section, can be used to estimate $V(t)^\top x$. Of course, our ultimate goal is to estimate x. This will be achieved, but is harder.

Observe from (III.1) that $\eta(t) = \dot{z}_1(t)$. Thus, one obtains (recalling that x is constant):

$$\dot{\eta}(t) = -\alpha\eta(t) + \dot{y}(t)^\top(y(t) - x). \tag{IV.1}$$

Similarly,

$$\dot{m}(t) = -\alpha m(t) + \dot{y}(t)^\top y(t); \tag{IV.2}$$

$$\dot{V}(t) = -\alpha V(t) + \dot{y}(t). \tag{IV.3}$$

$$\frac{d}{dt}\left[\eta(t) - m(t) + V^\top(t)x\right] = -\alpha\left[\eta(t) - m(t) + V^\top(t)x\right], \tag{IV.4}$$

which implies the following Lemma.

Lemma 4.1: For all t_0 and $t \geq t_0$, there holds:

$$\eta(t) - m(t) + V^\top(t)x = \left[\eta(t_0) - m(t_0) + V^\top(t_0)x\right]e^{-\alpha(t-t_0)}. \tag{IV.5}$$

As foreshadowed in the last section, we now present a lemma that shows that the agent located at $y(t)$ moves to a trajectory maintaining a constant distance d from the *estimated* position of the agent at position x.

Lemma 4.2: Consider (III.6) under Assumption 3.1. Suppose there exists a $\delta > 0$ such that in (III.5) $\hat{D}^2(0) > \delta$. Then $\hat{D}^2(t)$ converges exponentially to d^2, and there holds for all $t \geq 0$

$$\hat{D}^2(t) > \min\{\delta, d^2\} \tag{IV.6}$$

Proof: Because of (III.10) one obtains that,

$$\begin{aligned} \frac{d}{dt}\left\{\hat{D}^2(t) - d^2\right\} &= 2(\dot{y}(t) - \dot{\hat{x}}(t))^\top (y(t) - \hat{x}(t)) \\ &= -2(\hat{D}^2(t) - d^2)\hat{D}^2(t). \end{aligned} \tag{IV.7}$$

Observe that $\hat{D}$ is bounded and continuous. Consider first the case where $\delta > d^2$. Then the derivative above is initially negative, i.e. $\hat{D}^2(t)$ declines in value. By its continuity for $\hat{D}^2(t)$ to become less d^2, at some point it must equal d^2, when $\hat{D}(t)$ will stop changing. Since throughout this time $\hat{D}^2(t) \geq d^2$, convergence of $\hat{D}^2(t)$ to d^2 occurs at an exponential rate and $\hat{D}^2(t) \geq d^2$ for all t. On the other hand if $\delta \leq d^2$, then $\hat{D}^2(t) > \delta$ for all t, as the derivative of $\hat{D}^2(t)$ is nonnegative. Again exponential convergence of $\hat{D}^2(t)$ to d^2 occurs. ■

Now define $\tilde{x}(t) = \hat{x}(t) - x$. We have the following Lemma, which is the first of two aimed at establishing a Lyapunov function with certain properties for the overall system.

Lemma 4.3: Consider the system defined in (II.1) - (III.6), subject to the requirement that $\hat{D}(0) > 0$ and Assumption 3.1. Define:

$$\begin{aligned} L(t) = &\frac{1}{4\alpha}\left(\eta(t) - m(t) + V^\top(t)x\right)^2 + \frac{1}{2\gamma}\tilde{x}^\top(t)\tilde{x}(t) \\ &+ \frac{1}{4}\left(\hat{D}^2(t) - d^2\right)^2 + \sum_{i=1}^{3} L_i(t), \end{aligned} \tag{IV.8}$$

where $L_1(t) = (z_1(0)e^{-\alpha t})^2$, $L_2(t) = (z_2(0)e^{-\alpha t})^2$, and $L_3(t) = \left\|(z_3(0)e^{-\alpha t})^2\right\|^2$. Then, whenever $L(t)$ is bounded so also are the state variables $\hat{x}(t)$, $y(t)$, $z_i(t)$, $\eta(t)$, $m(t)$, and $V(t)$.

Proof: The second and third terms on the right hand side (IV.8) ensure the boundedness of $\hat{x}(t)$, $y(t)$ and $D(t)$. Thus as $L_i(t)$ are bounded (III.1), (III.2) and (III.3) ensure that the $z_i(t)$ are bounded as well. ■

We now define, for $\Delta > 0$ the set

$$S(\Delta) = \{[\hat{x}^\top, y^\top, z_1, z_2, z_3^\top]^\top | L \leq \Delta\}. \tag{IV.9}$$

Because of Lemma 4.3, the set $S(\Delta)$ is compact.

Next we identify an invariant set for (II.1) - (III.4).

Lemma 4.4: Consider the system defined in (II.1) - (III.6). Define the set $\mathcal{S}_I$ as the set of vectors $[\hat{x}^\top, y^\top, z_1, z_2, z_3^\top]^\top$ satisfying, $\hat{x} = x$, $\|y - x\| = d$, and $z_1 - z_2 + z_3^\top x = \frac{x^\top x}{2\alpha}$. Then $[\hat{x}^\top(0), y^\top(0), z_1(0), z_2(0), z_3^\top(0)]^\top \in \mathcal{S}_I$ implies $[\hat{x}^\top(t), y^\top(t), z_1(t), z_2(t), z_3^\top(t)]^\top \in \mathcal{S}_I$ for all $t \geq 0$.

Proof: First observe that on $\mathcal{S}_I$, $D = \hat{D} = d$. Thus on $\mathcal{S}_I$,

$$D^2 - y^\top y + 2y^\top \hat{x} = d^2 - y^\top y + 2x^\top y = x^\top x.$$

Further, from (III.1), (III.2) and (III.3) on $\mathcal{S}_I$

$$\begin{aligned} \dot{z}_1(t) - \dot{z}_2(t) + \dot{z}_3^\top(t)\hat{x}(t) &= -\alpha\left(z_1(t) - z_2(t) + z_3^\top(t)\hat{x}(t)\right) \\ &\quad + \frac{D^2(t) - y^\top(t)y(t) + 2y^\top(t)\hat{x}(t)}{2} \\ &= -\alpha\left(z_1(t) - z_2(t) + z_3^\top(t)x\right) \\ &\quad + \frac{x^\top x}{2} \end{aligned}$$

Consequently, on $\mathcal{S}_I$; $\dot{z}_1(t) - \dot{z}_2(t) + \dot{z}_3^\top(t)\hat{x}(t) = 0$, i.e. $\dot{z}_1(t) - \dot{z}_2(t) + \dot{z}_3^\top(t)x = 0$. Then,

$$\eta(t) - m(t) + V^\top(t)\hat{x}(t) = \dot{z}_1(t) - \dot{z}_2(t) + \dot{z}_3^\top(t)\hat{x}(t) = 0,$$

and so $\dot{\hat{x}}(t) = 0$ from (III.4). Last because of (IV.7), and the fact that $\hat{D} = d$, we have that $\dot{\hat{D}}(t) = 0$. ■

Then we have the following theorem whose proof is given in Appendix A.

Theorem 4.1: Consider the system defined in (II.1) - (III.6) and $\mathcal{S}_I$ defined in Lemma 4.4. Then for arbitrary initial conditions, subject to the requirement that $\hat{D}(0) > 0$ and Assumption 3.1, $\|\hat{x}^\top(t), y^\top(t), z_1(t), z_2(t), z_3^\top(t)\|$ is bounded $\forall t \geq 0$, and there holds:

$$\lim_{t\to\infty} \min_{z\in\mathcal{S}_I} \|z - [\hat{x}^\top(t), y^\top(t), z_1(t), z_2(t), z_3^\top(t)]^\top\| = 0. \tag{IV.10}$$

Further, convergence is uniform in the initial time.

Remark 4.1: Theorem 4.1 in particular implies that, $\tilde{x}(t)$ is bounded, $\lim_{t\to\infty} \tilde{x}(t) = 0$, $\lim_{t\to\infty} D(t) = d$, and $\lim_{t\to\infty} (y(t) - x - y^*(t)) = 0$, where $y^*(t)$ is a nonzero solution of (III.8) with nonzero $y^*(0)$ that for all t obeys $\|y^*(t)\| = d$. The first two limits follow from the definition of $\mathcal{S}_I$. The last limit is a consequence of (III.6).

V. Exponential convergence and Persistent Excitation

Having shown uniform asymptotic stability in the foregoing, we will now demonstrate that in fact the stability is exponential. For this we establish a persistent spanning condition first on $\dot{y}(t)$, and ultimately on $V(t)$. This will be the key to establishing the *exponential* convergence of $\hat{x}$ to x, which is a strengthening of the result of Theorem 4.1. First we establish certain conditions on the state transition matrix for $A(t)$, i.e. on $\Phi(t, t_0)$ that obeys for all t, t_0,

$$\dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0). \tag{V.1}$$

Lemma 5.1: Consider $\Phi(t, t_0)$ defined in (V.1),with Assumption 3.1. Then for all t, t_0

$$\Phi^\top(t, t_0)\Phi(t, t_0) = I.$$

Proof: Consider (III.8). Then for all t, t_0, and $y^*(t_0)$

$$y^*(t) = \Phi(t, t_0)y^*(t_0).$$

Further because of (III.10),

$$\|y^*(t)\| = \|y^*(t_0)\|.$$

Since this holds for all $y^*(t_0)$, the result holds. ∎

Next the first promised result.

Lemma 5.2: Consider the system defined in (II.1) - (III.6), subject to the requirement that $\hat{D}(0) > 0$ and Assumption 3.1. Then there exists a T_2 such that for all $t \geq 0$,

$$\frac{\alpha_1 d^2}{2} I \leq \int_t^{t+T_2} \dot{y}(\tau)\dot{y}(\tau)^\top d\tau \leq \alpha_2 I \tag{V.2}$$

Proof: See Appendix B ∎

We now show that $V(t)$ satisfies a p.e. condition as well.

Theorem 5.1: Consider the system defined in (II.1) - (III.6), subject to the requirement that $\hat{D}(0) > 0$ and Assumption 3.1. Then there exist $\alpha_3 > 0, \alpha_4 > 0, T_3 > 0$ such that for all $t \geq 0$

$$\alpha_3 I \leq \int_t^{t+T_3} V(\tau)V^\top(\tau) d\tau \leq \alpha_4 I. \tag{V.3}$$

Proof: Follows directly Lemma 5.2 and [6]. If we permitted $A(t)$ to lose differentiability on a countable set then the marriage of [6] with techniques developed in [8] will provide a proof. ∎

We are now ready to prove exponential converegence of $\hat{x}$ to x. Then in view of Lemma 4.2, and fact that $y(t)$ is bounded, $D(t)$ converges exponentially to d. Define $p(t) = \eta(t) - m(t) + V^\top(t)x$, and observe from (IV.5) that $\dot{p}(t) = -\alpha p(t)$. Observe also that $\eta(t) - m(t) + V^\top(t)\hat{x}(t) = p(t) + V^\top(t)\tilde{x}(t)$. Thus one obtains:

$$\begin{bmatrix} \dot{\tilde{x}}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} -\gamma V(t)V^\top(t) & -\gamma V(t) \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \tilde{x}(t) \\ p(t) \end{bmatrix}. \tag{V.4}$$

Theorem 5.2: Consider the system defined in (II.1) - (III.6), subject to the requirement that $\hat{D}(0) > 0$ and Assumption 3.1. Then (V.4) is eas.

Proof: From Theorem 5.1, (V.3) holds. Consequently, from [9], the system

$$\dot{\zeta} = -\gamma V V^\top \zeta$$

is eas. Then the triangular nature of (V.4) establishes the result. ∎

VI. Choosing $A(t)$

In this section we focus on selection of $A(t)$ to satisfy Assumption 3.1. Consider first $n = 2$, we show that with E as in (III.11) the matrix $A(t) = aE$ obeys the requirements of assumption 3.1. Indeed consider the Lemma below.

Lemma 6.1: With E as in (III.11), and a real scalar nonzero a consider

$$\dot{\xi}(t) = aE\xi(t), \tag{VI.1}$$

with $\xi : \mathbb{R} \to \mathbb{R}^2$. Denote $\xi = [\xi_1, \xi_2]^\top$. Define β as

$$\beta(t_0) = \angle(\xi_1 + \boldsymbol{i}\xi_2). \tag{VI.2}$$

i.e. the argument of the complex number $\xi_1 + \boldsymbol{i}\xi_2$. Then there holds for all $t \geq t_0 \geq 0$,

$$\xi(t) = \|\xi(t_0)\|[\cos(a(t-t_0)+\beta(t_0)), \sin(a(t-t_0)+\beta(t_0))]^\top. \tag{VI.3}$$

Proof: Follows from the facts that $\xi(t_0) = \|\xi(t_0)\|[\cos(\beta(t_0)), \sin(\beta(t_0))]^\top$, and that the state transition matrix corresponding to (VI.1) is:

$$e^{aEt} = \begin{bmatrix} \cos at & -\sin at \\ \sin at & \cos at \end{bmatrix}.$$

∎

The fact that (VI.3) satisfies (III.9) with y^* identified with ξ, is trivial to check. It is also clear that under this selection, $y(t)$ circumnavigates x with an angular speed of $|a|$.

In preparation for treating the $n = 3$ case, we make the following observations.

Lemma 6.2: Consider (VI.3). Suppose for any t_0, all $t \in \left[t_0, t_0 + \frac{\pi}{2|a|}\right]$, some $\theta \in \mathbb{R}^2$, there exists ϵ such that $|\theta^\top \dot{\xi}(t)| \leq \epsilon\|\theta\|$. Then

$$\epsilon \geq |a|\|\xi(t_0)\|\|\theta\|. \tag{VI.4}$$

Further with $\xi = [\xi_1, \xi_2]^\top$, for all t_0 and $i \in \{1, 2\}$,

$$\left|\xi_i\left(t_0 + \frac{\pi}{2|a|}\right) - \xi_i(t_0)\right| = \|\xi(t_0)\|. \tag{VI.5}$$

Proof: For some real ψ there holds $\theta = \|\theta\|[\sin\psi, -\cos\psi]^\top$. Hence, $E^\top\theta = \|\theta\|[\cos\psi, \sin\psi]^\top$. Thus under (VI.3),

$$\theta^\top E\xi(t) = \|\xi(t_0)\|\|\theta\|\cos(a(t-t_0) + \beta(t_0) - \psi). \tag{VI.6}$$

Therefore, on any interval $[t_0, t_0 + \pi/2|a|]$ the maximum of $|\theta^\top\dot{\xi}| = |a\theta^\top E\xi|$ is $|a|||\xi(t_0)||||\theta||$. Further (VI.5) is a direct consequence of (VI.3). ∎

Finally we prove the following lemma.

Lemma 6.3: Consider

$$\dot{\xi}(t) = af(t)E\xi(t), \tag{VI.7}$$

where $f : \mathbb{R} \to \mathbb{R}$ and $|f(t)| \leq 1 \quad \forall t$. Then for all $0 \leq t_0 \leq t$, $\|\xi(t) - \xi(t_0)\| \leq (t - t_0)|a|\,\|z(t_0)\|$.

Proof: Under (VI.7), for all $t \geq t_0$, $\|\xi(t)\| = \|\xi(t_0)\|$. Thus,

$$\begin{aligned} \|\xi(t) - \xi(t_0)\| &= \|\int_{t_0}^t f(\tau)aE\xi(\tau)d\tau\| \\ &\leq (t-t_0)|a|\,\|\xi(t_0)\| \end{aligned}$$

∎

To address the $n = 3$ case we first preclude the possibility that $A(t)$ can be a constant matrix. Indeed observe that no skew-symmetric matrix in $\mathbb{R}^{3\times 3}$ can be nonsingular, as if λ is an eigenvalue of a skew symmetric matrix then so is $-\lambda$. Thus for any odd n, an $n \times n$ skew symmetric matrix must have a zero eigenvalue. To complete the argument we present the following Lemma.

Lemma 6.4: Suppose in (III.8) $A(t) \equiv A$ for all t and A is singular. Then (III.9) cannot hold.

Proof: If A is singular, then e^{At} has an eigenvalue at one. Thus there exists a $y^*(0)$ such that for all t, $y^*(t) = e^{At}y^*(0)$ is a constant, i.e. for this $y^*(0)$, $\dot{y}^* \equiv 0$. ∎

Thus, we must search for a periodic $A(t)$ to meet the requirements of Assumption 3.1. Effectively, the $A(t)$ we will choose will switch periodically between the two 3×3 matrices

$$B = 0 \oplus (bE), \text{ and} \tag{VI.8}$$

$$C = (cE) \oplus 0 \tag{VI.9}$$

b and c being real nonzero scalars, and $\oplus$ denoting direct sum. However to ensure that the resulting matrix is differentiable, we require a differentiable transition between B and C. To achieve this define a nondecreasing $g : \mathbb{R} \to \mathbb{R}$, that obeys:

$$g(t) = 0 \quad \forall\, t \le 0 \tag{VI.10}$$

$$g(t) = 1, \quad \forall\, t \ge 1, \text{ and} \tag{VI.11}$$

$$g(t) \text{ is twice differentiable } \forall t. \tag{VI.12}$$

An example of such a $g(t)$ is

$$g(t) = \begin{cases} \frac{1}{2}\left(1 - \cos\left(\pi t\right)\right) & 0 \le t \le 1 \\ 0 & t < 0 \\ 1 & t > 1. \end{cases} \tag{VI.13}$$

Clearly this satisfies (VI.10) and (VI.11). Further (VI.12) holds as

$$\lim_{t \to 0_+} \dot{g}(t) = \lim_{t \to 1_-} \dot{g}(t) = 0.$$

Now, for nonzero scalars b and c, we will select $A(t)$ as follows. For a suitably small $\rho > 0$, define

$$\bar{T}_1 = \rho, \;\; \bar{T}_2 = \rho + \frac{\pi}{|b|}, \;\; \bar{T}_3 = 2\rho + \frac{\pi}{|b|}, \tag{VI.14}$$

and

$$\bar{T}_4 = 3\rho + \frac{\pi}{|b|}, \;\; \bar{T}_5 = 3\rho + \frac{\pi}{|b|} + \frac{\pi}{|c|}, \;\; T = \bar{T}_6 = 4\rho + \frac{\pi}{|b|} + \frac{\pi}{|c|}. \tag{VI.15}$$

For all t, let $K_T(t)$ denote the largest integer k satisfying $t \ge kT$ and let $r_T(t) = t - K_T(t)T$. Then define $A(t)$ as

$$A(t) = \begin{cases} g\left(\frac{t}{\rho}\right) B & 0 \le r_T(t) \le \bar{T}_1 \\ B & \bar{T}_1 \le r_T(t) \le \bar{T}_2 \\ \left(1 - g\left(\frac{t - \bar{T}_2}{\rho}\right)\right) B & \bar{T}_2 \le r_T(t) \le \bar{T}_3 \\ g\left(\frac{t - \bar{T}_3}{\rho}\right) C & \bar{T}_3 \le r_T(t) \le \bar{T}_4 \\ C & \bar{T}_4 \le r_T(t) \le \bar{T}_5 \\ \left(1 - g\left(\frac{t - \bar{T}_5}{\rho}\right)\right) C & \bar{T}_5 \le r_T(t) \le \bar{T}_6 = T \end{cases} \tag{VI.16}$$

Observe that (VI.16) automatically satisfies (i-iii) of Assumption 3.1. To show that it satisfies (iv) as well, we present the following result from [10].

Lemma 6.5: Suppose on a closed interval $\mathcal{I} \subset \mathbb{R}$ of length Ω, a signal $w : \mathcal{I} \to \mathbb{R}$ is twice differentiable and for some ϵ_1 and M'

$$|w(t)| \le \epsilon_1 \text{ and } |\ddot{w}(t)| \le M' \quad \forall \quad t \in \mathcal{I}.$$

Then for some M independent of ϵ_1, $\mathcal{I}$ and M', and $M'' = \max(M', 2\epsilon_1\Omega^{-2})$ one has:

$$|\dot{w}(t)| \le M(M''\epsilon_1)^{1/2} \quad \forall \quad t \in \mathcal{I}.$$

Next, we establish the following result.

Theorem 6.1: Consider (III.8) with $A(t)$ defined in (VI.14)-(VI.16). Then for every pair of nonzero b, c there exists a ρ^* such that (III.9) holds for all $0 < \rho \le \rho^*$.

Proof: See Appendix C. ∎

VII. Simulations

In this case we study the behaviour of the system in four different scenarios in 2-dimensional space.

In the first simulation we study the case where $x = [0.5,\ 3]^\top$, $d = 2$, and $y(0) = [8,\ 5]^\top$. The corresponding result is depicted in Fig. 1. A closer look at the agent trajectory reveals a very small radius turn near the point $[2,\ 1]^\top$. The reason for this behaviour is the following. The term $(\hat{D}^2(t) - d^2)(y(t) - \hat{x}(t))$ in (III.6) is designed to force $y(t)$ to move on a straight line trajectory in a manner that drives $\hat{D}$ to d. The second term $A(t)(y(t) - \hat{x}(t))$ forces $y(t)$ to rotate around $\hat{x}(t)$. Initially the first term is dominant, and the agent quickly travels a long distance on an almost straight line. By the time the agent reaches $[2, 1]^T$, the rotational motion component becomes comparable to the straight line motion component; hence the effect of this change shows itself as a sharp turn. To make the trajectory smoother, in the second scenario we consider the same setting with the difference that instead of using $\dot{y}(t)$, we use the normalized signal; in other words the agent is moving with constant speed. We use the normalized version of (III.6). Furhermore,

$$\dot{\bar{y}}(t) = \dot{\hat{x}}(t) - \left[(\hat{D}^2(t) - d^2)I - A(t)\right](y(t) - \hat{x}(t)),$$

$$\dot{y}(t) = \begin{cases} \dot{\bar{y}}(t)/\|\dot{\bar{y}}(t)\| & \|\dot{\bar{y}}(t)\| \neq 0 \\ 0 & \|\dot{\bar{y}}(t)\| = 0 \end{cases}$$

As can be observed, the small radius turn is replaced by one of larger radius. The result is presented in Fig. 2. In the first two cases the desired orbit at the prescribed distance is achieved. In the third simulation we studied the behaviour of the system when the source slowly drifts on a circle. on a circle with angular velocity equal to 0.005. See Fig. 3. The agent maintain its distance from the source in a neighbourhood of the desired distance. Notice that the speed of the source is always much less than the speed of the agent. In the last simulation we consider the case where the distance measurement is noisy, and it is assumed that $ln\, \bar{D} = ln\, D + \mu(t)$, where $\bar{D}$ is measurement and μ is a strict-sense stationary random process with $\mu(t) \sim N(0, \sigma^2)$, $\forall t$. The simulation result associated with this scenario is depicted in Fig. 4. As it can be observed the control law is still successful in moving the agent to an orbit with distance to the source kept close to its desired value.

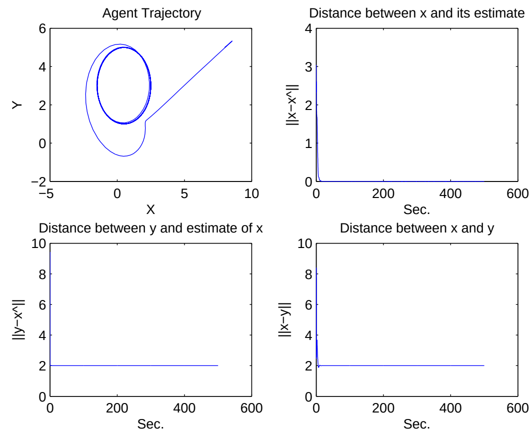

Fig. 1. Agent trajectory, agent distance from the estimate, agent distance from the real value, and distance between the estimate and true position of source.

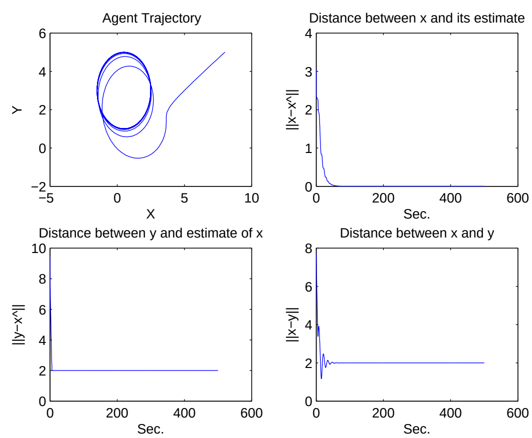

Fig. 2. Agent trajectory, agent distance from the estimate, agent distance from the real value, and distance between the estimate and true position of source, where agent is moving with constant speed.

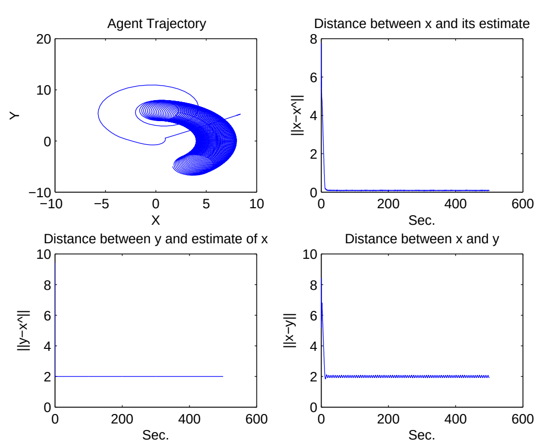

Fig. 3. Agent trajectory, agent distance from the estimate, agent distance from the real value, and distance between the estimate and true position of source, where the source is undergoing a drift.

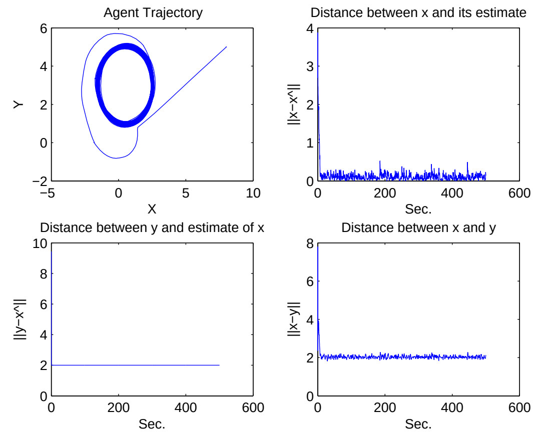

Fig. 4. Agent trajectory, agent distance from the estimate, agent distance from the real value, and distance between the estimate and true position of source, in the presence of noisy measurement.

VIII. Future Work and Concluding Remarks

In this paper we proposed an algorithm to solve the problem of monitoring a source at an unknown position by a single agent while the only information available to the agent is its distance to the target. Stability of the system has been established. Furthermore, in simulations the performance of the method in the presence of noise and in the situations where the source is undergoing a drifting motion is presented. An important future work is to establish stability of the system when the source is undergoing a drifting motion. Another possible extension of the current scheme is to consider the cases where more than one agent is present.

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Appendix

A. Proof of Theorem 4.1

First observe that the system (II.1) - (III.6) is in fact *periodic*. Thus, convergence if it holds, will be uniform in the initial time.

Consider $L(t)$ and $L_i(t)$ defined in Lemma 4.3. Now for every finite initial conditions there is a Δ such that

$$[\hat{x}^\top(0), y^\top(0), z_1(0), z_2(0), z_3^\top(0)]^\top \in S(\Delta).$$

As shown above this set is compact.

Note that for all $i \in \{1,2,3\}$, $\dot{L}_i(t) = -2\alpha L_i(t)$.

Then because of Lemma 4.2, (III.4), (III.6) and (IV.4), there holds

$$\begin{aligned}\dot{L}(t) = &-\frac{1}{2}\left(\eta(t) - m(t) + V^\top(t)x\right)^2\\ &- \tilde{x}^\top(t)V(t)\left(\eta(t) - m(t) + V^\top(t)x + V^\top(t)\tilde{x}(t)\right)\\ &+ \left(\hat{D}^2(t) - d^2\right)(y^\top(t) - \hat{x}^\top(t))\left(\dot{y}(t) - \dot{\hat{x}}(t)\right)\\ &- \alpha\sum_{i=1}^{3} L_i(t)\\ &= -\frac{1}{2}\left(\eta(t) - m(t) + V^\top(t)\hat{x}(t)\right)^2 - \frac{1}{2}(\tilde{x}^\top(t)V(t))^2\\ &- \left(\hat{D}^2(t) - d^2\right)^2\hat{D}^2(t) - \alpha\sum_{i=1}^{3} L_i(t) \le 0\end{aligned} \tag{A.1}$$

Thus all trajectories commencing in $S(\Delta)$ lie in $S(\Delta)$, and hence $\|\hat{x}^\top(t),\ y^\top(t),\ z_1(t),\ z_2(t),\ z_3^\top(t)\|$ is bounded $\forall t \ge 0$.

To prove the theorem we simply need to show that convergence occurs to $S(\Delta)\bigcap\mathcal{S}_I$. By Lasalle's theorem this will hold if $S(\Delta)\bigcap\mathcal{S}_I$ is the largest invariant set in $S(\Delta)$. Lemma 4.4 and (A.1) already shows that $S(\Delta)\bigcap\mathcal{S}_I$ is an invariant set. To show that it is the largest invariant set we must show that

$$\dot{L} \equiv 0, \tag{A.2}$$

implies that there hold for all $t \ge 0$

$$\hat{x}(t) = x, \tag{A.3}$$

$$\|y(t) - x\| = d, \tag{A.4}$$

$$z_1(t) - z_2(t) + z_3^\top(t)x = \frac{x^\top x}{2\alpha}. \tag{A.5}$$

Now (A.2) necessitates the pair of identities:

$$\dot{\hat{x}} \equiv 0, \tag{A.6}$$

$$\hat{D} \equiv d. \tag{A.7}$$

From (A.6) one obtains that for some constant x^*

$$\hat{x} \equiv x^*. \tag{A.8}$$

Further under (A.7), (A.8) and (A.6), (III.6) reduces to

$$\dot{y}(t) = A(t)(y(t) - x^*). \tag{A.9}$$

Thus, along trajectories corresponding to (A.2), there holds:

$$y(t) = x^* + y^*(t), \tag{A.10}$$

where $y^*(t)$ is a solution of (III.8). Denoting $a = \eta(t_0) - m(t_0) + V^\top(t_0)x$, because of (III.4), (A.6), (A.8), and Lemma 4.1 for all $t \ge t_0$, one has that:

$$\begin{aligned}0 &= \eta(t) - m(t) + V^\top(t)x^*\\ &= \eta(t) - m(t) + V^\top(t)x + V^\top(t)(x^* - x)\\ &= ae^{-\alpha(t-t_0)} + V^\top(t)(x^* - x).\end{aligned}$$

Thus one has

$$V^\top(t)(x^* - x) = -ae^{-\alpha(t-t_0)} \tag{A.11}$$

$$\frac{d}{dt}\left\{V^\top(t)(x^* - x)\right\} = \alpha ae^{-\alpha(t-t_0)}. \tag{A.12}$$

As $(x^* - x)$ is a constant, combining (A.11), (A.12) with (IV.3), we obtain that along trajectories corresponding to (A.2), for all $t \ge t_0$,

$$\alpha ae^{-\alpha(t-t_0)} = \alpha ae^{-\alpha(t-t_0)} + \dot{y}^\top(t)(x^* - x),$$

i.e. for all $t \ge 0$, $\dot{y}^\top(t)(x^* - x) = 0$. Given that $(x^* - x)$ is a constant and because of (III.9) and (A.10) this can only hold if $x^* = x$. Further in view of (A.7) along (A.2) $D \equiv d$.

Thus (A.3) and (A.4) are necessary for (A.2) to hold. Finally we observe from (A.2), (A.3), and the first term in (A.1) that:

$$\begin{aligned}&\eta - m + V^\top\hat{x} \equiv \eta - m + V^\top x \equiv 0,\\ \Rightarrow&\dot{\eta} - \dot{m} + \dot{V}^\top x \equiv 0\\ \Leftrightarrow& -\alpha(z_1 - z_2 + z_3^\top x) + \frac{D^2 - \|y\|^2 + 2y^\top x}{2} \equiv 0,\\ \Leftrightarrow& -\alpha\left(z_1 - z_2 + z_3^\top x - \frac{x^\top x}{2\alpha}\right) \equiv 0,\ \text{as}\ D \equiv d,\end{aligned}$$

establishing (A.5).

B. Proof of Lemma 5.2

A consequence of assumption 3.1 is that for all unit $\theta \in \mathbb{R}^n$, $t \ge 0$ and $z \in \mathbb{R}^n$, there holds:

$$\alpha_1\|z\|^2 \le \int_t^{t+T_1} |\theta^\top A(\tau)\Phi(\tau,t)z|^2 d\tau. \tag{B.1}$$

Further because of (III.6) for all $t_1 \ge 0$ and $t \ge t_1$ there holds:

$$\begin{aligned}\dot{y}(t) - \dot{\hat{x}}(t) = &A(t)\Phi(t,t_1)(y(t_1) - \hat{x}(t_1))\\ &- A(t)\int_{t_1}^{t}\Phi(t,\tau)(\hat{D}^2(\tau) - d^2)(y(\tau) - \hat{x}(\tau))d\tau.\end{aligned} \tag{B.2}$$

Assumption 3.1 ensures that $A(t)$ is bounded. Thus there exists M_2 such that

$$\|A(t)\| \le M_2 \quad \forall t. \tag{B.3}$$

Further because of Lemma 4.2 and Theorem 4.1, there is a $\gamma > 0$, such that for every $\epsilon > 0$, there is a t_2 such that for all $t \ge t_2$,

$$|d - \|y(t) - \hat{x}(t)\|| \le \epsilon, \tag{B.4}$$

$$\|\dot{\hat{x}}(t)\| \leq \epsilon, \tag{B.5}$$

$$|\hat{D}^2(t) - d^2| \leq \epsilon e^{-\gamma(t-t_2)}. \tag{B.6}$$

Thus because of Lemma 5.1, (B.2)- (B.6) for every unit $\theta \in \mathbb{R}^n$ and $t \geq t_2$

$$|\theta^\top \dot{y}(t)| \geq |\theta^\top A(t)\Phi(t,t_2)(y(t_2) - \hat{x}(t_2))| - \epsilon - \frac{\epsilon(d+\epsilon)M_2}{\gamma}.$$

Thus there exist K_i all positive such that for all $t \geq t_2$ there holds

$$|\theta^\top \dot{y}(t)|^2 \geq |\theta^\top A(t)\Phi(t,t_2)(y(t_2) - \hat{x}(t_2))|^2 - \sum_{i=1}^{4} K_i \epsilon^i. \tag{B.7}$$

Choose $T_2 = T_1 + t_2$. Then because of (B.1) and (B.7) for all $t > 0$, there holds:

$$\begin{aligned}
&\int_t^{t+T_2} |\theta^\top \dot{y}(\tau)|^2 d\tau \\
&\geq \int_{t+t_2}^{t+T_2} |\theta^\top \dot{y}(\tau)|^2 d\tau \\
&\geq \int_{t+t_2}^{t+T_2} |\theta^\top A(\tau)\Phi(\tau,t_2)(y(t_2) - \hat{x}(t_2))|^2 d\tau \\
&-T_1 \textstyle\sum_{i=1}^4 K_i \epsilon^i \\
&\geq \alpha_1 (d-\epsilon)^2 - T_1 \textstyle\sum_{i=1}^4 K_i \epsilon^i \\
&\geq \alpha_1 d^2 - 2\alpha_1 d\epsilon - T_1 \textstyle\sum_{i=1}^4 K_i \epsilon^i
\end{aligned}$$

Then the left inequality in (V.2) follows by choosing ϵ so that

$$T_1 \sum_{i=1}^{4} K_i \epsilon^i + 2\alpha_1 d\epsilon \leq \alpha_1 d^2/2. \tag{B.8}$$

The right inequality in (V.2) follows from the boundedness of $\hat{x}$, (III.6), (III.8), (III.9), and Lemma 4.1.

C. *Proof of Theorem 6.1*

We will prove the result by contradiction. First observe that as $A(t)$ is differentiable and $\ddot{y}^*$ is bounded. Also observe that if (III.9) holds for $\|y^*(0)\| = 1$, then it holds for arbitrary $\|y^*(0)\|$. Thus assume that $\|y^*(0)\| = 1$. Consequently for all $t \geq 0$

$$\|y^*(t)\| = 1. \tag{C.1}$$

Suppose (III.9) is violated. Then for all $\epsilon_2 > 0$ and $T_3 > 0$, there exists a t_0 and a unit norm $\theta = [\theta_1, \theta_2, \theta_3]^\top \in \mathbb{R}^3$, such that

$$\int_{t_0}^{t_0+T_3} (\theta^\top \dot{y}^*(\tau))^2 d\tau \leq \epsilon_2^2.$$

Thus from Lemma 6.5 for some M_3, all $\epsilon_2 > 0$, some $T_4(\epsilon_2)$, dependent only on the bound on $\ddot{y}^*(\cdot)$ and ϵ_2, and all $T_3 > T_4(\epsilon_2)$, there exists a t_0 and unit norm $\theta \in \mathbb{R}^3$, for which

$$\left|\theta^\top \dot{y}^*(t)\right| \leq M_3 \epsilon_2^{1/2} \quad \forall t \in [t_0, t_0 + T_3]. \tag{C.2}$$

Choose

$$t_1 = \min\{kT \geq t_0 + T_4(\epsilon_2) | k \in \mathbb{Z}_+\}. \tag{C.3}$$

Denote $y^* = [y_1^*, y_2^*, y_3^*]^\top$. Observe that at least one of $\|[\theta_1, \theta_2]^\top\|$ or $\|[\theta_2, \theta_3]^\top\|$ must exceed $1/\sqrt{3}$, since θ has unit norm. We consider two cases.

Case I: $\|[\theta_1, \theta_2]^\top\| > 1/\sqrt{3}$ **.**

Since the inequality in (C.2) holds on the indicated interval, it must hold for all $t \in [t_1 + kT + \bar{T}_4, t_1 + kT + \bar{T}_5]$, $k \in \mathbb{Z}$.

Thus for all $t \in [t_1 + kT + \bar{T}_4, t_1 + kT + \bar{T}_5]$ and $k \in \mathbb{Z}$, there holds:

$$|[\theta_1, \theta_2]^\top [\dot{y}_1^*(t), \dot{y}_2^*(t)]| \leq M_3 \epsilon_2^{1/2}. \tag{C.4}$$

Now for all $t \in [t_1 + kT + \bar{T}_4, t_1 + kT + \bar{T}_5]$, there also holds:

$$\begin{bmatrix} \dot{y}_1^*(t) \\ \dot{y}_2^*(t) \end{bmatrix} = cE \begin{bmatrix} y_1^*(t) \\ y_2^*(t) \end{bmatrix}.$$

Thus, from (VI.4) of Lemma 6.2 and the hypothesis of the case, we obtain that for all $k \in \mathbb{Z}$,

$$\|[y_1^*(t_1 + kT + \bar{T}_4), y_2^*(t_1 + kT + \bar{T}_4)]^\top\| \leq \frac{\sqrt{3}M_3}{|c|} \epsilon_2^{1/2}. \tag{C.5}$$

Further with some $h_1 : \mathbb{R} \to \mathbb{R}$, in the interval $[kT + \bar{T}_4, (k+1)T]$, $\begin{bmatrix} \dot{y}_1^*(t) \\ \dot{y}_2^*(t) \end{bmatrix} = h_1(t)E \begin{bmatrix} y_1^*(t) \\ y_2^*(t) \end{bmatrix}$. Thus,

$$\begin{aligned}
&\|[y_1^*(t_1 + (k+1)T), y_2^*(t_1 + (k+1)T)]^\top\| = \\
&\|[y_1^*(t_1 + kT + \bar{T}_4), y_2^*(t_1 + kT + \bar{T}_4)]^\top\| \leq \tfrac{\sqrt{3}M_3}{|c|} \epsilon_2^{1/2}.
\end{aligned} \tag{C.6}$$

Consequently because of (C.1), there holds:

$$\begin{aligned}
&\|[y_2^*(t_1 + (k+1)T), y_3^*(t_1 + (k+1)T)]^\top\| \geq \\
&|y_3^*(t_1 + (k+1)T)| \geq 1 - \tfrac{\sqrt{3}M_3}{|c|} \epsilon_2^{1/2}
\end{aligned} \tag{C.7}$$

Further throughout the interval $t \in [t_1 + kT, t_1 + kT + T_3]$ for some $h_2 : \mathbb{R} \to \mathbb{R}$, $|h_2(t)| \leq 1$,

$$\begin{bmatrix} \dot{y}_2^*(t) \\ \dot{y}_3^*(t) \end{bmatrix} = h_2(t)E \begin{bmatrix} y_2^*(t) \\ y_3^*(t) \end{bmatrix}. \tag{C.8}$$

Thus from Lemma 6.3 and (C.6)

$$|y_2^*(t_1 + kT + \bar{T}_1)| \leq \frac{\sqrt{3}M_3}{|c|} \epsilon_2^{1/2} + \rho|b|. \tag{C.9}$$

Also from (C.8) and (C.7)

$$\|[y_2^*(t), y_3^*(t)]^\top\| \geq 1 - \frac{\sqrt{3}M_3}{|c|} \epsilon_2^{1/2}$$

holds for all $t \in [t_1 + kT, t_1 + kT + \bar{T}_3]$. Notice in the interval $[[t_1 + kT + \bar{T}_1, t_1 + kT + \bar{T}_2]$, (C.8) holds with $h_2(t) = b$. Thus from (VI.5) of Lemma 6.2,

$$\begin{aligned}
&|y_2^*(t_1 + kT + \bar{T}_2)| + |y_2^*(t_1 + kT + \bar{T}_1)| \\
&\geq |y_2^*(t_1 + T + \bar{T}_2) - y_2^*(t_1 + T + \bar{T}_1)| \geq 1 - \tfrac{\sqrt{3}M_3}{|c|} \epsilon_2^{1/2}
\end{aligned} \tag{C.10}$$

Consequently, from (C.9)

$$|y_2^*(t_1 + kT + \bar{T}_2)| \geq 1 - \frac{2\sqrt{3}M_3}{|c|} \epsilon_2^{1/2} - \rho|b|.$$

Further, from Lemma 6.3

$$|y_2^*(t_1 + kT + \bar{T}_4)| \geq 1 - \frac{2\sqrt{3}M_3}{|c|} \epsilon_2^{1/2} - \rho\left(2|b| + |c|\right). \tag{C.11}$$

Then for

$$\rho < \frac{1}{4\left(|b| + |c|\right)}. \tag{C.12}$$

and sufficiently small ϵ_2, (C.11), contradicts with (C.5).

Case II: $\|[\theta_2, \theta_3]^\top\| > 1/\sqrt{3}$ **.** Follows similarly with the same set of ρ given in (C.12).